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The Inverse Stefan Problem as a Problem of Nonlinear Approximation Theory

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1. INTRODUCTION

Consider the following one-dimensional one-phase Stefan problem:

$$Lu \equiv u_{xx} - u_t = 0 \quad \text{in} \quad 0 < x < s(t), \quad 0 < t \leqslant T, \tag{1.1}$$

$$u(x, 0) = f(x), \quad 0 \leqslant x \leqslant b = s(0) > 0, \tag{1.2}$$

$$u_x(0, t) = g(t), \quad 0 < t \leqslant T, \tag{1.3}$$

$$u(s(t), t) = 0, \quad 0 < t \leqslant T, \tag{1.4}$$

$$u_x(s(t), t) = -\dot{s}(t), \quad 0 < t \leqslant T, \tag{1.5}$$

where to a given pair of data functions f, g one searches the free boundary s and the solution u satisfying the system (1.1)–(1.5). Physically, f can be interpreted as the initial temperature distribution, g as the time dependent heat flux, s as the melting interface and u as the temperature of the liquid phase.

Suppose, conversely, a given initial distribution f and a prescribed interface $\tilde{s}$ with $\tilde{s}(0) = b$. The problem is to find a function g which together with f generates a free boundary s which coincides with, or, more realistically, which best approximates the given boundary $\tilde{s}$ in a sense to be specified later. This associated problem of optimal control is usually called the inverse Stefan problem (abbreviated: ISP).

Problems of this type have been considered by various authors and several numerical procedures were suggested for the construction of the unknown g [1, 2, 4, 5, 19, 20]. It is well known [18] that the ISP is not properly posed in the sense of Hadamard, i.e., the function g does not depend continuously on the data function $\tilde{s}$. Hence, numerical procedures are usually based on solving ill conditioned linear or nonlinear equations which have to be regular-

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ized by imposing additional restrictions on g which are suggested by the physical background [1, 4, 19]. As to the numerical computation of g great efforts are required in order to avoid the accumulation of rounding errors. Moreover, only the uniqueness of a solution g of the ISP is known [4, 9], but not its existence.

These facts suggest the formulation of the ISP as a nonlinear approximation problem, which does not require the existence of a solution to the ISP to be known and which can be solved by a highly stable iterative Newton-like procedure developed by Osborne and Watson [17]. The approximation problem is the following:

Find a $g^* \in R$ with

$$\| S(g^*) - \tilde{s} \| = \inf\{\| S(g) - \tilde{s} \| \mid g \in R\}, \tag{1.6}$$

where $S: g \mapsto s$ is the solution operator of (1.1)–(1.5), R represents a set of admissible controls and $\| \cdot \|$ is some norm on $C[0, T]$. In addition to the advantages mentioned above, this formulation of the ISP allows one to introduce additional restrictions on g (specified by R) in a natural way and always yields a best approximation $s^* = S(g^*)$ with respect to the given set R.

The purpose of this paper is to investigate the existence of a solution g^* of (1.6) under certain conditions on R and, in the third section, the characterization of an optimal solution $s^* = S(g^*)$ for the special norm $\| \cdot \| = \| \cdot \|_2$. In a subsequent paper we shall treat the numerical aspects of (1.6) for the L_2- and the L_∞-norms; we shall present the numerical algorithms and we treat questions of convergence and numerical stability.

2. Existence of a Best Approximation

The question of existence cannot easily be answered in the general setting (1.6). Hence, we initially restrict our considerations to the numerically important case of a finite dimensional (in a certain sense maximal) set R and to the uniform norm

$$\| s \| = \sup_{0 \leqslant t \leqslant T} | s(t) |.$$

At the end of this section we shall discuss questions and problems appearing in the infinite dimensional case.

Let V be a finite dimensional subspace of $C[0, T]$ and consider the set $R := \{v \in V \mid v \leqslant 0\}$ (relations between functions are to be understood for each argument in their common domain of definition). R is the maximal subset of V for which the existence of a solution of (1.1)–(1.5) is known under certain conditions on f indicated below [6]. Our method of proof will be such

that for any nonempty closed subset $R' \subset R$, we shall obtain, as a simple consequence, the existence of an optimal g^* with respect to R'.
We make the following assumptions on the initial function f:

$$f \in C^1[0, b], \qquad f \geqslant 0, \qquad f(b) = 0. \tag{2.1}$$

It is well known that under these conditions there exists to each $g \in R$ a unique solution (u, s) of the free boundary problem (1.1)–(1.5), where $u \in C(\bar{\Omega}(s))$, u_t, $u_{xx} \in C(\Omega(s))$, $s \in C^1[0, T] \cap C^\infty(0, T]$ and $\Omega(s) = \{(x, t) \mid 0 < x < s(t), 0 < t \leqslant T\}$ (see [6]). Thus the solution operator

$$S: R \to C[0, T], \qquad g \mapsto s,$$

is well defined. It is also Lipschitz continuous and s depends monotonically on g and is a monotonically nondecreasing function [6, Theorems 5, 6, 5]. Now we can state the main result of this section.

THEOREM 2.2. *Let $\tilde{s} \in C[0, T]$, $\tilde{s}(0) = b$, R and V as defined above. Then there exists an optimal $g^* \in R$, i.e.,*

$$\| S(g^*) - \tilde{s} \| = \inf_{g \in R} \| S(g) - \tilde{s} \| =: \rho(\tilde{s}). \tag{2.3}$$

Proof. As in the case of linear approximation problems, we shall show that it is sufficient to take the infimum over a compact subset of R. Recalling the continuity of the error functional $e(g) = \| S(g) - \tilde{s} \|$ this will complete the proof.

Because of the trivial relation $\rho(\tilde{s}) \leqslant \| \tilde{s} - S(0) \|$ and the monotonicity of the boundaries $s = S(g)$ for $g \in R$, it is sufficient to take the infimum only over those $g \in R$ which generate boundaries $S(g)$ satisfying the inclusion:

$$\bigwedge_{t \in [0,T]} b \leqslant S(g)(t) \leqslant \| \tilde{s} \| + \| S(0) - \tilde{s} \| =: M. \tag{2.4}$$

In fact, each $s = S(g)$ not satisfying (2.4) is a worse approximation to $\tilde{s}$ than $S(0)$ because of the estimate: $\| s - \tilde{s} \| \geqslant \| s \| - \| \tilde{s} \| > \| S(0) - \tilde{s} \|$.
We shall show that all $g \in R$ satisfying (2.4) lie in a compact set $K \subset R$ which depends only on the constant M.

Every $s = S(g)$ has the integral representation

$$s(t) = b - \int_0^t g(\tau)\, d\tau - \int_0^{s(t)} u(x, t)\, dx + \int_0^b f(x)\, dx, \tag{2.5}$$

which can be easily obtained by integrating the identity $Lu = 0$ over $\Omega(s)$. Now fix a $g \in R$ satisfying (2.4) and cousider the solution w of the following boundary value problem:

$$\begin{aligned}
Lw &= 0 && \text{in } 0 < x < M, \quad 0 < t \leqslant T, \\
w(x, 0) &= f(x), && x \in [0, b], \\
&= 0, && x \in (b, M], \\
w_x(0, t) &= g(t), && w(M, t) = 0, \quad 0 < t \leqslant T.
\end{aligned} \tag{2.6}$$

By the maximum principle (MP) and the parabolic version of Hopf's lemma [10, Theorem 14, p. 49], $w \geqslant 0$ and, therefore, again by the MP $w \geqslant u$ in $\Omega(s)$. Thus

$$\begin{aligned}
s(t) &\geqslant b - \int_0^t g(\tau)\, d\tau - \int_0^{s(t)} w(x, t)\, dx + \int_0^b f(x)\, dx \\
&\geqslant b - \int_0^t g(\tau)\, d\tau - \int_0^M w(x, t)\, dx + \int_0^b f(x)\, dx.
\end{aligned} \tag{2.7}$$

Integrating $Lw = 0$ over the rectangle $[0, M] \times [0, t]$ gives:

$$\int_0^M w(x, t)\, dx = \int_0^b f(x)\, dx + \int_0^t w_x(M, \tau)\, d\tau - \int_0^t g(\tau)\, d\tau.$$

Inserting in (2.7) leads to: $s(t) \geqslant b - \int_0^t w_x(M, \tau)\, d\tau$. Combining this estimate with (2.4) and observing $w(M, t) = 0$ and $w \geqslant 0$ in its domain of definition, we finally get the inclusion:

$$\bigwedge_{0 < t \leqslant T} 0 \leqslant -\int_0^t w_x(M, \tau)\, d\tau \leqslant s(t) - b \leqslant M. \tag{2.8}$$

From this inequality we will obtain restrictions describing the compact subset $K \subset R$. To this end, we shall derive an integral equation for the function $w_x(M, t)$ by using the fundamental solution K of the heat equation and the associated Neumann function N:

$$\begin{aligned}
K(x, t, \xi, \tau) &= \frac{1}{2\pi^{1/2}(t - \tau)^{1/2}} \exp\left\{\frac{-(x - \xi)^2}{4(t - \tau)}\right\}, \\
N(x, t, \xi, \tau) &= K(x, t, \xi, \tau) + K(-x, t, \xi, \tau).
\end{aligned}$$

Integrating Green's identity with N and w over $[0, M] \times [\epsilon, t - \epsilon]$ and letting ϵ tend to zero one obtains the following representation of w (for details, see [8]):

$$w(x, t) = -\int_0^t N(x, t, 0, \tau)\, g(\tau)\, d\tau + \int_0^t N(x, t, M, \tau)\, w_x(M, \tau)\, d\tau$$
$$+ \int_0^b N(x, t, \xi, 0) f(\xi)\, d\xi.$$

Differentiating with respect to x, letting x tend to $M - 0$, and observing the fundamental jump relation [8, Lemma 1] we get the following integral equation for the function $w_x(M, t)$:

$$w_x(M, t) - 2\int_0^t N_x(M, t, M, \tau)\, w_x(M, \tau)\, d\tau$$
$$= -2\int_0^t N_x(M, t, 0, \tau)\, g(\tau)\, d\tau + 2\int_0^b N_x(M, t, \xi, 0) f(\xi)\, d\xi. \quad (2.9)$$

Because of $M > b$, the kernel of the last integral remains bounded for $t \to 0$. It is easily verified that all three kernels appearing in (2.9) are nonpositive and that the following estimate holds:

$$| N_x(M, t, M, \tau)| \leqslant \frac{M}{2\pi^{1/2}} \left(\frac{3}{2eM^2}\right)^{3/2} \leqslant \frac{1}{2\pi^{1/2}M^2}.$$

Inserting this in (2.9) and observing the signs of $w_x(M, t)$, g and f we have

$$0 \leqslant 2\int_0^t N_x(M, t, 0, \tau)\, g(\tau)\, d\tau \leqslant -w_x(M, t) - \frac{1}{\pi^{1/2}M^2}\int_0^t w_x(M, \tau)\, d\tau.$$

Integrating this inequality from 0 to t and using twice the right-hand side of (2.8) we conclude

$$0 \leqslant 2\int_0^t \int_0^r N_x(M, r, 0, \tau)\, g(\tau)\, d\tau\, dr \leqslant M + \frac{t}{\pi^{1/2}M}. \quad (2.10)$$

This inclusion holds for all $t \in [0, T]$. We shall show that these infinitely many restrictions on g describe a compact subset of R. Let n be the dimension of V, choose a basis $v_1, ..., v_n$ of V with $\| v_i \| = 1$ and write g as $\sum a_i v_i$. Then (2.10) can be written in the form

$$\bigwedge_{0 \leqslant t \leqslant T} 0 \leqslant \sum_{i=1}^{n} a_i \varphi_i(t) \leqslant M + \frac{t}{\pi^{1/2}M} \leqslant M', \quad (2.11)$$

where $\varphi_i(t) = 2\int_0^t \int_0^r N_x(M, r, 0, \tau) v_i(\tau)\, d\tau\, dr$ and $M' = M + T/\pi^{1/2}M$. Showing the linear independence of the φ_i would complete our proof. In fact, we then could choose n points $0 \leqslant t_1 < \cdots < t_n \leqslant T$ with $\det(\varphi_i(t_j)) \neq 0$ [7, p. 79], and, with the notations $\Phi = (\varphi_i(t_j))$, $d = a\Phi \in \mathbb{R}^n$ and $\|\cdot\|_\infty$ the discrete maximum norm, we obtain from (2.11): $\| d \|_\infty \leqslant M'$ and consequently

$$\| a \|_\infty = \| \Phi^{-1} d \|_\infty \leqslant \| \Phi^{-1} \|_\infty \| d \|_\infty \leqslant M' \| \Phi^{-1} \|_\infty := C.$$

Hence a is in the compact cube $\tilde{K} = \{x \in \mathbb{R}^n \mid \| x \|_\infty \leqslant C\}$ and therefore, $g \in K := \{v \in R \mid \| v \| \leqslant nC\}$, where the constant C depends only on M, T, and the special choice of the basis of V.

Now, in order to show the linear independence of the φ_i, suppose $\sum_{i=1}^n a_i \varphi_i = 0$. It follows: $\sum_{i=1}^n a_i \dot{\varphi}_i = 0$, i.e.,

$$\bigwedge_{0 \leqslant t \leqslant T} \int_0^t N_x(M, t, 0, \tau) \left[\sum_{i=1}^n a_i v_i(\tau) \right] d\tau = 0.$$

The function $z(x, t) = \int_0^t N_x(x, t, 0, \tau)\mu(\tau)\, d\tau$ is the solution of the heat equation in the strip $(0, \infty) \times (0, T]$ satisfying the boundary conditions $z(x, 0) = 0$, $z(0, t) = \mu(t)$. Now, $z(M, t) = 0$ for all $t \in [0, T]$ implies $z(x, t) = 0$ in the strip $[M, \infty) \times [0, T]$ and, by analytic continuation, $z(x, t) = 0$ in $[0, \infty) \times [0, T]$. It follows $\sum_{i=1}^n a_i v_i = 0$ and, by the linear independence of the v_i, $a_i = 0$, $i = 1, \ldots, n$. Hence, the φ_i are linearly independent and the theorem is proved. ∎

COROLLARY 2.12. *For each nonempty closed subset $R' \subset R$, there exists an optimal g^* with respect to R'.*

Proof. As R' is nonempty, take a $g_0 \in R'$ and replace the definition of the constant M in (2.4) by $M := \| \tilde{s} \| + \| S(g_0) - \tilde{s} \|$. Now the conclusion follows by the same arguments as in the proof of Theorem 2.2. ∎

In the following lemma we state a simple result concerning the quality $\rho(\tilde{s})$ of the best approximation $s^* = S(g^*)$. Let (V_n) be an ascending sequence of subspaces (i.e., $V_n \subset V_{n+1}$) the union of which is dense in $C[0, T]$ and denote by R_n the cones $\{v \in V_n \mid v \leqslant 0\}$. Suppose further that $\tilde{s}$ lies in the range of S, i.e., that there is a $\tilde{g}$, $\tilde{g} \leqslant 0$, with $S(\tilde{g}) = \tilde{s}$. Then, clearly,

$$\rho_n(\tilde{s}) = \inf_{g \in R_n} \| S(g) - \tilde{s} \|$$

converges to zero with $n \to \infty$, because of the continuity of S. If in addition an upper bound is known for the minimal distance between R_n and $\tilde{g}$ (a Jackson-type estimate), this statement can be given exactly.

LEMMA 2.13. *Suppose that there is a $\tilde{g} \in C[0, T]$, $\tilde{g} \leqslant 0$, with $\tilde{s} = S(\tilde{g})$ and assume the following estimate to hold*:

$$\inf\{\| \tilde{g} - v \| \mid v \in R_n\} \leqslant Cn^{-\alpha}, \tag{2.14}$$

where C and α are positive constants independent of n. Then $\rho_n(\tilde{s})$ is bounded by $TCn^{-\alpha}$.

Proof. Since R_n is a finite dimensional closed cone, there exists a best approximation $\hat{v}_n \in R_n$ to $\tilde{g}$. S is Lipschitz continuous with a Lipschitz constant not greater than the final time T [14, Theorem 5.1]. Denoting by g_n^* a solution of (2.3) we have

$$\begin{aligned}\rho_n(\tilde{s}) &= \| S(g_n^*) - \tilde{s} \| = \| S(g_n^*) - S(\tilde{g})\| \leqslant \| S(\hat{v}_n) - S(\tilde{g})\| \\ &\leqslant T \| \hat{v}_n - \tilde{g} \| \leqslant TCn^{-\alpha}. \quad \blacksquare\end{aligned}$$

There are numerous estimates of the kind of (2.14) with known constants C and α depending on the shape of $\tilde{g}$ for various subspaces $V_n \subset C[0, T]$ [16]. Thus Lemma (2.13) gives a satisfactory answer refering to the quality of the best approximation, provided the existence of $\tilde{g}$ is known a priori. Actually, instead of claiming the existence of a $\tilde{g}$ it would be sufficient to have $\tilde{s}$ in the closure of the range $Q = \{S(\tilde{g}) \mid g \in C[0, T], g \leqslant 0\}$.

As far as we know, however, the problem of determining the range Q of S (or of some suitable subset of it) is not yet solved. Only necessary conditions to be satisfied by an $\tilde{s} \in Q$, have recently been found by Kinderlehrer and Nirenberg [15]. Clearly, $\tilde{s} \in Q$ has to be a monotonically increasing C^{∞}-function for $t > 0$. The result of [15] is the additional growth condition:

$$| \tilde{s}^{(n)}(t)| \leqslant M(2n)!\, \gamma^{-n}, \qquad M > 0,\ \gamma > 0. \tag{2.15}$$

A family of functions satisfying such an estimate is called a *Gevrey class* [12]. For each function $\tilde{s}$ satisfying (2.15) the series

$$\tilde{u}(x, t) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial t^n} (x - \tilde{s}(t))^{2n} \tag{2.16}$$

converges in a neighbourhood of $\tilde{s}$ representing there the solution of the noncharacteristic Cauchy problem (1.1), (1.4), (1.5) [13]. The difficulty lies in the fact that this solution is uniquely determined even without any specification of the initial function f [9].

Another question not answered by the preceeding lemma which, however, is important for the physical application is that of convergence of the sequence (g_n^*) of optimal controls to the solution $\tilde{g}$ of the ISP (if existing). Regarding the ill-posedness of the ISP one would neither expect this sequence to converge not even to be bounded. On the other side, sign conditions ($g \leqslant 0$) usually exert a regularizing effect [3, 18] on the improperly posed problem. In the case of the ISP, however, the continuous dependence of the sign restricted solution $\tilde{g}$ on $\tilde{s}$ is not proved and is still to be investigated.

3. Characterization of an Optimal Boundary

In this section we shall restrict ourselves to the finite dimensional case and, therefore, suppress the subscript n. We consider an optimal boundary s^* with respect to a certain type of parameter space V ("Haar space") assuring the uniqueness of the optimal solution $s^* = S(g^*)$. We shall characterize such a s^* by an alternation property well known from the linear Tchebychev approximation theory. Let us first recall a property of S recently found by the author [14] which we shall need in Definition 3.4, below.

LEMMA 3.1. *Denote by B the Banach space $(C[0, T], \|\cdot\|)$ and let $A := \{g \in B \mid g \leqslant 0\}$. Further assume $f \in C^3[0, b]$ and $f''(b) = [f'(b)]^2$. Then the solution operator S: $A \to B$, $g \mapsto s$ is Lipschitz continuously Frechet differentiable and the F-derivative S'_g: $B \to B$, $h \mapsto \delta s$ is the linear solution operator of the following system*:

$$\delta s(t) = -\int_0^t h(\tau)\, d\tau - \int_0^{s(t)} z(x, t)\, dx, \qquad 0 \leqslant t \leqslant T, \tag{3.2}$$

$$\begin{aligned} Lz &= 0 && \text{in } \Omega(s), \text{ where } s = S(g),\\ z(x, 0) &= 0, && 0 \leqslant x \leqslant b,\\ z_x(0, t) &= h(t), && 0 < t \leqslant T,\\ z(s(t), t) &= \dot{s}(t)\, \delta s(t), && 0 < t \leqslant T. \end{aligned} \tag{3.3}$$

Proof. See [14, Theorems 4.1 and 4.14]. ∎

Now let $V \subset B$ be a finite dimensional subspace of dimension n and replace the definition of R by

$$R := \{g \in V \mid g < 0\}.$$

For in the following, we shall need an open (i.e., an only formal restricted) parameter set R. At the end of this section we shall discuss the difficulties appearing when R is closed.

DEFINITION 3.4. For each $g \in R$ we denote by $d(g)$ the dimension of the associated *tangential space* $S'_g(V)$. The nonlinear approximating family $S(R)$

is called a *global Haar set*, if for each $g \in R$ and all $h \in R$ the difference $S(g) - S(h)$ possesses at most $d(g) - 1$ distinct zeros or vanishes identically. $S(R)$ is said to form a *local Haar set* if for each $g \in R$ every nontrivial function $\delta s \in S_g'(V)$ possesses at most $d(g) - 1$ (distinct) zeros [16, p. 136].

For families of functions satisfying both the global and the local Haar conditions there are well-known results concerning uniqueness and behaviour of a best approximation s^* with respect to R (see [16]). As it will turn out, satisfaction of the Haar conditions depends on the usual Haar condition being fulfilled by the parameter space V. Roughly speaking, the difference $\Delta s = \bar{s} - s$ has at most as many zeros for $t > 0$ as the difference $h = \bar{g} - g$ of the corresponding controls. The precise statement will be given in Theorem 3.6, below. First we need an appropriate notion of the multiplicity of a zero.

DEFINITION 3.5. For a function $\varphi \in B$ with $\varphi(t_0) = 0$ the number

$$V(t_0) = 1 \qquad \text{if } \varphi \text{ changes its sign in } t_0,$$
$$= 2 \qquad \text{otherwise,}$$

is called the *multiplicity of the zero t_0 of φ*.

THEOREM 3.6. *Let $g, \bar{g} \in R$, $h = \bar{g} - g$, $s = S(g)$, $\bar{s} = S(\bar{g})$, and $\Delta s = \bar{s} - s$. Assume further that Δs has exactly K distinct zeros for $t > 0$*

$$0 < \tau_1 < \cdots < \tau_K \leqslant T \qquad \textit{with } m = \sum_{i=1}^{K} V(\tau_i).$$

Then there exist at least $m + 1$ numbers $0 = \gamma_0 < \cdots < \gamma_m$ and m zeros $0 < t_1 < \cdots < t_m$ of h with

$$\bigwedge_{1 \leqslant i \leqslant m} h(t_i) = 0, \qquad t_i \in (\gamma_{i-1}, \gamma_i),\ h \text{ changes sign in } (\gamma_{i-1}, \gamma_i). \tag{3.7}$$

Proof. If h changes sign infinitely often the theorem is proved. Thus we assume that h changes sign only a finite number of times.

Let $\alpha(t)$ be the continuous, piecewise differentiable function $\min(s(t), \bar{s}(t))$ and $w = \bar{u} - u$ in $\Omega(\alpha)$. We consider level curves $\Gamma_i \subset \Omega(\alpha)$ with $w \mid \Gamma_i = 0$ starting in $(s(\tau_i), \tau_i)$ and leading into $\Omega(\alpha)$. By a similar method, Friedman and Jensen [11] proved the convexity of the free boundary under certain conditions on f. We shall show that the Γ_i are differentiable, monotonically decreasing curves intersecting the t-axis at the points $(0, \gamma_i)$. To this aim we prove by induction the following more general proposition:

PROPOSITION 3.8. *For each zero τ_i of Δs with $V(\tau_i) = 1$, there exists a differentiable level curve Γ_j with $j > \sum_{l=1}^{i-1} V(\tau_l)$ and a number $\gamma_j > \gamma_{j-1}$ such that*

$$\Gamma_j = \Big\{(x, t) \in \Omega(\alpha) \mid x = \zeta(t), w(\zeta(t), t) = 0, \zeta(\tau_i) = s(\tau_i) = \bar{s}(\tau_i),$$

$$\zeta(\gamma_j) = 0, \bigwedge_{\gamma_j \leqslant t \leqslant \tau_i} \dot{\zeta}(t) \geqslant 0\Big\}. \tag{3.9}$$

If $V(\tau_i) = 2$, there exists another curve Γ_{j+1} besides Γ_j and a number $\gamma_{j+1} > \gamma_j$ with

$$\Gamma_{j+1} = \Big\{(x, t) \in \Omega(\alpha) \mid x = \varphi(t) < \zeta(t), w(\varphi(t), t) = 0, \varphi(\tau_i) = \zeta(\tau_i)$$

$$= s(\tau_i), \varphi(\gamma_{j+1}) = 0, \bigwedge_{\gamma_{j+1} \leqslant t \leqslant \tau_i} \dot{\varphi}(t) \geqslant 0\Big\}; \tag{3.10}$$

h changes sign in each interval (γ_{j-1}, γ_j) (respectively (γ_j, γ_{j+1})).

Proof of the proposition. $K = 1$.

Without loss of generality we assume that $\bar{s}(t) > s(t) = x(t)$ holds for $0 < t < \tau_1$, because otherwise, we could replace s by $\bar{s}$ and vice versa. Observing $\Delta s \mid (0, \tau_1) > 0$ and the monotone dependence of s on g, there is an $\epsilon > 0$ such that $h \mid [0, \epsilon] \leqslant 0$. The strong maximum principle (SMP, [10, p. 34]) and $w(s(t), t) > 0$ for $0 < t < \tau_1$ imply $w(x, \epsilon) > 0$ for all $x \in [0, s(\epsilon)]$. Now let K_r be an open ball with center $(s(\tau_1), \tau_1)$ and radius r and define

$$D_r := K_r \cap \Omega(\alpha) \cap \{\epsilon < t < \tau_1\}, \quad F_r := (\partial K_r \cap D_r) \cup (\{x = 0, t > \epsilon\} \cap D_r)$$

(F_r is the circular part of the parabolic boundary of ∂D_r united with a part of the t-axis if $r > s(\tau_1)$). Denote by P_r the lower boundary point of F_r.

ASSERTION A. *For each $r > 0$, the function w has a zero $N_r = (\xi_r, \eta_r) \in F_r$.*

Proof A. For all $t \in [\epsilon, \tau_1)$ we have by the SMP: $w(s(t), t) = \bar{u}(s(t), t) > 0$. Now assume $w \mid \bar{D}_r \geqslant 0$. Then, by the SMP, $w \mid D_r > 0$. Hence, $w(s(\tau_1), \tau_1) = (\bar{u} - u)(s(\tau_1), \tau_1) = 0$ is a minimum of w with respect to $\bar{D}_r$. By the parabolic version of Hopf's lemma we conclude

$$0 > w_x(s(\tau_1), \tau_1) = -\dot{\bar{s}}(\tau_1) + \dot{s}(\tau_1) = -\Delta \dot{s}(\tau_1) \geqslant 0.$$

Hence, our assumption on w was false, i.e., w has a negative minimum in $\bar{D}_r$, and this minimum must be attained on F_r. As $w \mid F_r$ is a continuous function and $w(P_r) > 0$, there must be a zero N_r of w on F_r.

For each $r > 0$ we denote by N_r the zero of w on F_r with minimal t-coordinate (the set of zeros on F_r is closed!).

ASSERTION B. *The zeros N_r as chosen above are located on a continuous differentiable level curve Γ_1 satisfying* (3.9).

Proof B. In $N_r = (\xi_r, \eta_r)$ the function w achieves its minimum with respect to the parabolic region $D_r' := D_r \cap \{t \leqslant \eta_r\}$. The SMP implies $w \mid D_r' > 0$, and, again by Hopf's lemma, we conclude $w_x(N_r) > 0$. Since w_x is continuous in $\Omega(\alpha)$, there is an open ball $U \subset \Omega(\alpha)$ with center N_r such that $w_x \mid U > 0$. Now, the function w_t/w_x is continuous in U and we may consider the ordinary initial value problem

$$\dot{\zeta}(t) = \frac{-w_t(\zeta(t), t)}{w_x(\zeta(t), t)}, \qquad \zeta(\eta_r) = \xi_r . \tag{3.11}$$

Choosing U sufficiently small, there is, by Peano's theorem, a continuously differentiable solution curve ζ of (3.11) defined in U with $w(\zeta(t), t) = 0$. In fact, we have

$$\frac{d}{dt} w(\zeta(t), t) = \dot{\zeta}(t)\, w_x(\zeta(t), t) + w_t(\zeta(t), t) = 0,$$

and $w(\zeta(\eta_r), \eta_r) = w(N_r) = 0$. Thus, $w(\zeta(t), t) = 0$ in U. Observing $w_t(N_r) \leqslant 0$ and $w_x(N_r) > 0$ we conclude

$$\dot{\zeta}(\eta_r) = \frac{-w_t(N_r)}{w_x(N_r)} \geqslant 0.$$

Now assume that there is a $\bar{r} > r$ with $F_{\bar{r}} \cap U \neq \varnothing$ and $N_{\bar{r}}$ not lying on $\{\zeta(t), t\}$. Then the solution $\bar{\zeta}$ of (3.11) corresponding to the initial point $N_{\bar{r}}$ intersects F_r below N_r which is a contradiction to the choice of N_r as having minimal t-coordinate. Thus, the curve ζ connects all zeros N_r and can be continued until $x = 0$ intersecting the t-axis in the point $(0, \gamma_1)$. The property $\zeta(\tau_1) = s(\tau_1) = \bar{s}(\tau_1)$ is obvious.

ASSERTION C. *The function h changes sign in $t_1 \in (\gamma_0, \gamma_1)$.*

Proof C. Suppose $h \mid [0, \gamma_1] \leqslant 0$. Then, by the SMP, $w \mid \Omega(\alpha) \cap \{t \leqslant \gamma_1\} > 0$, and, consequently, $0 = w(0, \gamma_1)$ is an absolute minimum of w with respect to the closure of $\Omega(\alpha) \cap \{t \leqslant \gamma_1\}$. Hopf's lemma yields: $h(\gamma_1) = w_x(0, \gamma_1) > 0$. This contradiction implies the existence of a $t_1 < \gamma_1$ with $h(t_1) = 0$ and h changing sign in t_1.

Note that the main tools in proving the existence of the level curve Γ_1 were the SMP and Hopf's lemma. The latter may be applied provided the "inside strong sphere condition" [10, p. 48] is satisfied, which, however, does not hold in $\Omega(\alpha)$ at the point $(\alpha(\tau_1), \tau_1)$, as $\dot{\alpha}$ is only piecewise continuous. But in case of one space variable, as was already stated by Sherman [21], it is sufficient to claim that $\dot{\alpha}$ and $w(\alpha(t), t)$ can be continued continuously for $t > \tau_1$. Actually, this implies the inside strong sphere property with respect to the modified region without changing the values of w for $t \leqslant \tau_1$.

ASSERTION D. *If* $V(\tau_1) = 2$, *there exists another level curve* Γ_2 *with the properties* (3.10).

Proof D. In proving Assertion A we used the inequality $\Delta\dot{s}(\tau_1) \leqslant 0$, which holds for simple and double zeros. From $V(\tau_1) = 2$ and hence, $\Delta\dot{s}(\tau_1) = 0$, we shall deduce the existence of a second curve Γ_2 satisfying (3.9). Let $D := \{(x, t) \in \Omega(\alpha) \mid 0 < x < \zeta(t), \gamma_1 < t \leqslant \tau_1\}$ and redefine

$$D_r := D \cap K_r, \qquad F_r := (\partial K_r \cap D_r) \cup (\{x = 0, t > \gamma_1\} \cap D_r),$$

and P_r to be the intersection point of ∂K_r and Γ_1, i.e., the lower boundary points of F_r. Assume $w \mid \bar{D}_r \leqslant 0$. Then by the SMP $w \mid D_r < 0$, because $w_x(\zeta(t), t) > 0$. Hence, w achieves its maximum in $(s(\tau_1), \tau_1)$ with respect to $\bar{D}_r$. It is easily verified that $\lim_{t\to\tau_1-0} \dot{\zeta}(t)$ exists and is equal to $\dot{\bar{s}}(\tau_1) + \dot{s}(\tau_1) = 2\dot{s}(\tau_1)$. Thus, we can apply Hopf's lemma which implies $w_x(s(\tau_1), \tau_1) > 0$. We get

$$0 = \Delta\dot{s}(\tau_1) = -\bar{u}_x(\bar{s}(\tau_1), \tau_1) + u_x(s(\tau_1), \tau_1) = -w_x(s(\tau_1), \tau_1) < 0.$$

Consequently, our assumption $w \mid \bar{D}_r \leqslant 0$ was wrong, and w must have a positive maximum on F_r. Since $w_x(\zeta(t), t) > 0$ and $w(\zeta(t), t) = 0$, there must be an open ball U with center P_r such that $w \mid F_r \cap U < 0$. The continuity of w on F_r implies the existence of a zero $M_r \in F_r - U$. As before, we now take M_r to be the zero of w on F_r with minimal t-coordinate. For each $r > 0$ it is $M_r \neq P_r$, because $w \mid F_r \cap U < 0$. By the arguments used in the proof of Assertion B, the points M_r, $r > 0$, must lie on a continuously differentiable curve Γ_2 satisfying (3.10). It remains to prove that h changes sign in the interval (γ_1, γ_2). Assume $h \mid [\gamma_1, \gamma_2] \geqslant 0$ and define $D' := D \cap \{t \leqslant \gamma_2\}$. It follows $w \mid \bar{D}' \leqslant 0$ and, by the SMP, $w \mid D' < 0$. Hence, $0 = w(0, \gamma_2)$ is an absolute maximum with respect to $\bar{D}'$ and, again by Hopf's lemma, we conclude $h(\gamma_2) = w_x(0, \gamma_2) < 0$.

This contradiction shows that there must be a $t_2 \in (\gamma_1, \gamma_2)$ with $h(t_2) = 0$ and h changing sign in (γ_1, γ_2).

This completes the proof of Proposition 3.8 for $K = 1$.

$K - 1 \to K$. Assume proposition 3.8 to be proved for all $i \leqslant K - 1$ and define $m' := \sum_{i=1}^{K-1} V(\tau_i)$. Then h has changed its sign at least m' times at the points $0 < t_1 < \cdots < t_{m'}$. The situation is now the same as in the case $K = 1$ except only that the x-axis has to be replaced by the level curve $\Gamma_{m'}$. Hence, the induction step can be performed using exactly the same arguments as in the proof for $K = 1$. ∎

Remark 3.12.

(a) The method of proving the existence of a second level curve Γ_2 (Assertion D) cannot be repeated infinitely many times: already the second curve Γ_2 has a horizontal tangent at the point $(s(\tau_1), \tau_1)$, i.e., $\lim_{t\to\tau_1} \dot{\varphi}(t) = \infty$, and hence, φ cannot be continued continuously for $t > \tau_1$. Thus, the application of Hopf's lemma is not permitted.

(b) Assume $\Delta s \mid [\tau_1, \tau_2] = 0$. Then for $t \in [\tau_1, \tau_2]$ it follows: $w(s(t), t) = w_x(s(t), t) = 0$, and the uniqueness of the solution of the noncharacteristic Cauchy problem implies $h \mid [\tau_1, \tau_2] = 0$.

The next theorem contains an analogous statement for the pair $(z, \delta s)$ defining the F-derivative of S.

THEOREM 3.13. *Let be $g \in A$, $s = S(g)$, and for a $h \in B$, $\delta s = S_g'h$. Assume that δs has exactly K distinct zeros for $t > 0$*

$$0 < \tau_1 < \cdots < \tau_K \leqslant T \qquad \text{with } m = \sum_{i=1}^{K} V(\tau_i).$$

Then there exist at least $m + 1$ numbers $0 = \gamma_0 < \cdots < \gamma_m$ and m zeros $0 < t_1 < \cdots < t_m$ of h satisfying (3.7).

Proof. By Lemma 3.1, for a given $h \in B$, the pair $(z, \delta s)$ is the unique solution of (3.2), (3.3). In [14, Theorem 3.8] it is proved that δs is continuously differentiable and satisfies the initial value problem

$$\delta\dot{s}(t) = -\dot{s}^2(t)\,\delta s(t) - z_x(s(t), t), \qquad \delta s(0) = 0. \tag{3.14}$$

Hence, for each zero τ_i of δs we have $\delta\dot{s}(\tau_i) = -z_x(s(\tau_i), \tau_i)$. Replace now in the proof of Theorem 3.6, the functions α, Δs, and w by s, δs, and z, respectively. Then the conclusion follows by exactly the same arguments. ∎

DEFINITION 3.14. For $g \in V$ the tangential space $S_g'(V)$ is said to be *regular* if $\dim(V) = \dim(S_g'(V))$.

COROLLARY 3.15. *For each $g \in V$ the tangential space $S_g'(V)$ is regular.*

Proof. Replace Δs by δs in Remark 3.12b. Then $\delta s \equiv 0$ implies $h \equiv 0$, i.e., S_g' is one-to-one and, consequently, $\dim(V) = \dim(S_g'(V))$. ∎

Theorems 3.6 and 3.13 essentially state that the functions Δs and δs change sign at most as many times as the corresponding variation h. Thus, for $t > 0$, Δs and δs have at most as many zeros as the corresponding h. But assuming V to be a Haar space, the always existing additional zero $\tau_0 = 0$ of both Δs and δs would prevent $S(R)$ from being a global or local Haar set, because Δs and δs might have too many zeros.

Thus we shall exclude the zero $\tau_0 = 0$ by considering the following "truncated problem." Let $\epsilon \in (0, T)$ be fixed, define

$$S_\epsilon : A \to C[\epsilon, T], \qquad g \mapsto s = S(g) \mid [\epsilon, T],$$

$$S'_{\epsilon,g} : B \to C[\epsilon, T], \qquad h \mapsto \delta s = S'_g h \mid [\epsilon, T],$$

and find a $g_\epsilon^* \in R$ with

$$\| S_\epsilon(g^*) - \tilde{s} \|_\epsilon = \inf_{g \in R} \| S_\epsilon(g) - \tilde{s} \|_\epsilon =: \rho_\epsilon(\tilde{s}), \tag{3.16}$$

where $\| s \|_\epsilon = \sup_{\epsilon \leqslant t \leqslant T} | s(t) |$.

LEMMA 3.17. *The operator S_ϵ: $A \to (C[\epsilon, T], \| \cdot \|_\epsilon)$ is Lipschitz continuously Frechet differentiable and has the derivative $S'_{\epsilon,g}$.*

Proof. This is an obvious consequence of Lemma 3.1.

LEMMA 3.18. *If V is a Haar space with* $\dim(V) = n$, *then*

$$\bigwedge_{g \in R} d_\epsilon(g) := \dim(S'_{\epsilon,g}(V)) = n, \tag{3.19}$$

and $S_\epsilon(R)$ is a global and local Haar set.

Proof. To prove (3.19), let $\delta s \mid [\epsilon, T] = 0$. Then by Remark 3.12b, with Δs replaced by δs, we conclude $h \mid [\epsilon, T] = 0$ and further, by Haar's condition in V, $h \equiv 0$. Thus $S'_{\epsilon,g}$ is one-to-one on V, which implies (3.19). The global and local Haar conditions follow from Theorems (3.6) and (3.13), respectively. ∎

Now we are able to characterize best ϵ-approximations $s_\epsilon^* = S_\epsilon(g_\epsilon^*)$ by means of well-known theorems about nonlinear Haar sets [16].

LEMMA 3.20. *Let $\tilde{s} \in B$, $\tilde{s}(0) = b$, and an $\epsilon \in (0, T)$ be given and assume V to be a Haar space of dimension n. Then the following statements hold*:

(a) *A function $s_\epsilon^* = S_\epsilon(g_\epsilon^*)$ is a solution of* (3.16), *if and only if there is a* *set of* $n+1$ *alternation points, i.e., there are points* $\epsilon \leqslant t_1 < \cdots < t_{n+1} \leqslant T$ *such that the error function* $e_\epsilon := s_\epsilon^* - \tilde{s}$ *satisfies*

$$\begin{aligned} &\bigwedge_{1\leqslant i\leqslant n+1} |e_\epsilon(t_i)| = \|e_\epsilon\|_\epsilon, \\ &\bigwedge_{1\leqslant i\leqslant n} e_\epsilon(t_i) + e_\epsilon(t_{i+1}) = 0. \end{aligned} \tag{3.21}$$

(b) *There is at most one best approximation* $s_\epsilon^* = S_\epsilon(g_\epsilon^*)$ *of* $\tilde{s}$.

Proof. Lemma 3.18 supplies all hypotheses for Theorems 86 and 87 in [16] stating precisely the assertions (a) and (b). ∎

It remains to investigate the behaviour of the mapping $p: \epsilon \mapsto s_\epsilon^*$ for $\epsilon \to 0$. It will turn out that there is a $\delta > 0$ below which p is constant.

THEOREM 3.22. *If* V *is a Haar space of dimension* n *the following statements hold*:

(a) *A function* $s^* = S(g^*)$ *is a solution of* (2.3), *if and only if there is a set of* $n+1$ *alternation points, i.e., there are points* $0 < t_1 < \cdots < t_{n+1} \leqslant T$ *such that the error function* $e := s^* - \tilde{s}$ *satisfies* (3.21) *without subscript* ϵ.

(b) *There is at most one best approximation* $s^* = S(g^*)$.

Proof. If $\rho(\tilde{s}) = 0$, the assertion is true. Thus we assume $\rho(\tilde{s}) > 0$. Define

$$R^* := \{g \in R \mid \|S(g) - \tilde{s}\| = \rho(\tilde{s})\},$$

and observe that R^* is closed. From the proof of the existence theorem 2.2 it follows that R^* is bounded and, therefore, compact. For an arbitrary $g \in R$ we have the estimate [6]

$$\left\| \frac{d}{dt} S(g) \right\| \leqslant \max\left(\sup_{0\leqslant x\leqslant b} |f'(x)|, \|g\|\right).$$

Hence, there is a constant $N > 0$ such that

$$\sup \left\{ \left\| \frac{d}{dt} S(g) \right\| \;\middle|\; g \in R^* \right\} \leqslant N.$$

Denote by $\omega(\tilde{s}; \delta)$ the modulus of continuity of $\tilde{s}$ and choose a $\delta > 0$ sufficiently small such that $\omega(\tilde{s}; \delta) < \frac{1}{2}\rho(\tilde{s})$ and $\delta < \rho(\tilde{s})/2N$. Then we have for $0 < \epsilon \leqslant \delta$ and all $s \in S(R^*)$

$$\begin{aligned} \sup_{0\leqslant t\leqslant \epsilon} |s(t) - \tilde{s}(t)| &\leqslant b + \epsilon\|\dot{s}\| - b + \omega(\tilde{s}; \epsilon) \leqslant N\epsilon + \omega(\tilde{s}; \epsilon) \\ &\leqslant N\delta + \omega(\tilde{s}; \delta) < \tfrac{1}{2}\rho(\tilde{s}) + \tfrac{1}{2}\rho(\tilde{s}) = \rho(\tilde{s}). \end{aligned}$$

Hence, all $s \in S(R^*)$ are also best approximations on the smaller intervals $[\epsilon, T]$ with $0 < \epsilon \leqslant \delta$, i.e., with respect to the approximation problem (3.16). Since by Lemma 3.20 these best approximations are uniquely determined, the set $S(R^*)$ contains at most one element s^*. Thus assertion (b) is proved. Assertion (a) follows from (3.21), since for all $0 < \epsilon < \delta$, s^* is also the unique solution of (3.16). ∎

Note that the crucial assumption on R to be open cannot be omitted. In fact, with respect to the closed set R used in Section 2, the preceding characterization theorem would be generally false. For the points t_i, where g^+ touches the restricting zero-function, ought to be contained in a "generalized alternation set" introduced in linear approximation theory [22] for restricted approximation problems. To transfer these ideas to the nonlinear approximation problems. To transfer these ideas to the nonlinear approximation problem (2.3), it would be necessary to count the touching points and the alternation points in an appropriate way yet to be found.

At the end of this section we quote a result of great practical importance which applies whenever the best approximation s^* is calculated numerically by an iterative procedure to be described in a subsequent paper. This "inclusion theorem" is analogous to the result of De la Vallee–Poussin which is well known in linear approximation theory.

THEOREM 3.23. *Let V be a Haar space of dimension n and assume a given $s \in S(R)$ and $n + 1$ points $0 < t_1 < \cdots < t_{n+1} \leqslant T$ with*

$$\bigwedge_{1 \leqslant i \leqslant n+1} \tilde{s}(t_i) - s(t_i) \neq 0,$$
$$\bigwedge_{1 \leqslant i \leqslant n} \operatorname{sgn}(\tilde{s}(t_i) - s(t_i)) = -\operatorname{sgn}(\tilde{s}(t_{i+1}) - s(t_{i+1})). \tag{3.24}$$

Then the following inclusion holds:

$$\min_{1 \leqslant i \leqslant n+1} |\tilde{s}(t_i) - s(t_i)| \leqslant \rho(\tilde{s}) \leqslant \|\tilde{s} - s\|. \tag{3.25}$$

Proof. The right-hand inequality is trivial. For the proof of the left side, recall that for $\epsilon > 0$ $S_\epsilon(R)$ is a global Haar set (Lemma 3.18). An application of Theorem 88 in [16] yields for all $0 < \epsilon \leqslant t_1$

$$\min_{1 \leqslant i \leqslant n+1} |\tilde{s}(t_i) - s(t_i)| \leqslant \rho_\epsilon(\tilde{s}).$$

The obvious relation $\rho_\epsilon(\tilde{s}) \leqslant \rho(\tilde{s})$ completes the proof.

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